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The Burnside ring of fusion systems ☆

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Abstract

Given a saturated fusion system $\mathcal{F}$ on a finite p-group S we define a ring $\mathcal{A}(\mathcal{F})$ modeled on the Burnside ring $\mathcal{A}(G)$ of finite groups. We show that these rings have several properties in common. When $\mathcal{F}$ is the fusion system of G we describe the relationship between these rings.

Keywords: Fusion system; p-Local finite group; Orbit category; Burnside ring

1. Introduction

Let G be a finite group. The category of finite G-sets is closed under formation of disjoint unions $X \sqcup Y$ and products $X \times Y$. The set of isomorphism classes of finite G-sets therefore forms a commutative monoid under the operation $\sqcup$. Its Grothendieck group completion is denoted $\mathcal{A}(G)$. Disjoint unions (coproducts) of G-sets distribute over products whence products of G-sets render $\mathcal{A}(G)$ a commutative ring. This is the Burnside ring of G, and it is one of the fundamental representation rings of G (see [1] for a survey on the subject).

A finite group G and a choice of a Sylow p-subgroup S in G give rise to a fusion system $\mathcal{F}_S(G)$ over S. This is a small category whose objects are the subgroups of S and it contains all the p-local information of G. The more general concept of a "saturated fusion system $\mathcal{F}$ on

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a finite p-group S" was introduced by Lluis Puig (e.g. in [9]). It will be recalled in Section 2. A saturated fusion system $\mathcal{F}$ over S is associated with an orbit category $\mathcal{O}(\mathcal{F})$, see e.g. [2], [9, §4] and Definition 2.5. Even when $\mathcal{F}$ is the fusion system of a finite group G, $\mathcal{O}(\mathcal{F})$ is very different from $\mathcal{O}_G$ or any of its subcategories.

Let $\mathcal{O}(\mathcal{F}^c)$ be the full subcategory of $\mathcal{O}(\mathcal{F})$ generated by the $\mathcal{F}$-centric subgroups. The additive extension of $\mathcal{O}(\mathcal{F}^c)$, denoted $\mathcal{O}(\mathcal{F}^c)_\sqcup$ was defined in [8]. Informally, this is the category of finite sequences in $\mathcal{O}(\mathcal{F}^c)$. See Section 2 for more details. Concatenation of sequences is the categorical coproduct in $\mathcal{O}(\mathcal{F}^c)_\sqcup$. Puig proves in [9, Proposition 4.7] that $\mathcal{O}(\mathcal{F}^c)_\sqcup$ has products (in the category-theory sense). This product distributes over the coproduct. Therefore we can define the Burnside ring $\mathcal{A}(\mathcal{F})$ as the group completion of the abelian monoid of the isomorphism classes of the objects of $\mathcal{O}(\mathcal{F}^c)_\sqcup$ under coproducts. The product in $\mathcal{A}(\mathcal{F})$ is induced from the product in $\mathcal{O}(\mathcal{F}^c)_\sqcup$. We will prove that $\mathcal{A}(\mathcal{F})$ has similar properties to the Burnside ring of a finite group.

1.1. Theorem. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then $\mathbb{Q} \otimes \mathcal{A}(\mathcal{F})$ is a commutative semisimple $\mathbb{Q}$-algebra with one primitive idempotent for every isomorphism class of objects of $\mathcal{O}(\mathcal{F}^c)$.*

A more elaborate version of this result will be proven in Theorem 3.3.

Now let G be a finite group. It is easy to see that the subgroup of $\mathcal{A}(G)$ generated by the G-sets all of whose isotropy groups are finite p-groups forms an ideal $\mathcal{A}(G; p)$ in $\mathcal{A}(G)$. We will show this in Section 3. The identity in $\mathcal{A}(G)$ corresponds to the G-set with one element. This G-set is not present in $\mathcal{A}(G; p)$ and in general this subring of $\mathcal{A}(G)$ does not have a unit. Given a saturated fusion system let $\mathcal{A}(\mathcal{F})_{(p)}$ denote $\mathbb{Z}_{(p)} \otimes \mathcal{A}(\mathcal{F})$. We will prove in Corollary 3.5:

1.2. Theorem. *For any saturated fusion system $\mathcal{F}$ the ring $\mathcal{A}(\mathcal{F})_{(p)}$ has a unit.*

The ring $\mathcal{A}(G; p)$ is not local in general as it exhibits $\mathbb{F}_q$ as a quotient ring for several primes q (see [6]). However, we will give a description of the prime spectrum of $\mathcal{A}(\mathcal{F})_{(p)}$ in Corollary 3.10 which in particular includes the following statement.

1.3. Theorem. *For any saturated fusion system $\mathcal{F}$ the ring $\mathcal{A}(\mathcal{F})_{(p)}$ is a local ring.*

When $\mathcal{F}$ is the fusion system of a finite group G we expect to find a relationship between the Burnside rings $\mathcal{A}(G)$ and $\mathcal{A}(\mathcal{F})$. To do this we now define a certain section of the ring $\mathcal{A}(G)$.

Let $\mathcal{A}(G; p\text{-}\neg\text{cent})$ denote the subgroup of $\mathcal{A}(G)$ generated by the G-sets all of whose isotropy groups are p-subgroups which are not p-centric subgroups of G (see Section 2 below). We will see that this is an ideal of $\mathcal{A}(G)$ which is contained in $\mathcal{A}(G; p)$. The quotient ring $\mathcal{A}(G; p)/\mathcal{A}(G, p\text{-}\neg\text{cent})$ is denoted $\mathcal{A}^{p\text{-cent}}(G)$. We shall write $\mathcal{A}^{p\text{-cent}}(G)_{(p)}$ for $\mathbb{Z}_{(p)} \otimes \mathcal{A}^{p\text{-cent}}(G)$. In Theorem 3.11 we will prove

1.4. Theorem. *Let S be a Sylow p-subgroup of a finite group G and let $\mathcal{F}$ denote the associated fusion system. Then the rings $\mathcal{A}(\mathcal{F})_{(p)}$ and $\mathcal{A}^{p\text{-cent}}(G)_{(p)}$ are isomorphic.*

1.5. Notation. If X, Y are objects in a category $\mathcal{C}$ we denote the set of morphisms $X \to Y$ by $\mathcal{C}(X, Y)$. When $\mathcal{C}$ is equal to a fusion system $\mathcal{F}$ or to its orbit category $\mathcal{O}(\mathcal{F})$, we will also use the standard notation $\mathrm{Hom}_{\mathcal{F}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(X, Y)$ which is widespread in the literature.

Organization of the paper

In Section 2, we introduce saturated fusion systems and we define the associated Burnside ring. We also set up some notation and prove some basic results. In Section 3, we consider the analogue of the "table of marks". Then we study both the rational and p-local versions of the Burnside ring. We conclude this section by analyzing the relation to the classical Burnside ring for saturated fusion systems induced by finite groups. In Section 4, we compute some examples. In particular, we describe the Burnside rings of the Ruiz–Viruel saturated fusion systems [11].

2. Saturated fusion systems and their Burnside ring

A fusion system $\mathcal{F}$ on a finite p-group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$, where $P, Q \leqslant S$, consist of group monomorphisms which satisfy the following two conditions:

(a) The set $\mathrm{Hom}_S(P, Q)$ of all the homomorphisms $P \to Q$ which are induced by conjugation by elements of S is contained in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$. In particular all the inclusions $P \leqslant Q$ are morphisms in $\mathcal{F}$.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

It easily follows that $\varphi : P \to Q$ in $\mathcal{F}$ is an isomorphism in $\mathcal{F}$ if and only if it is an isomorphism of groups. In this case we say that P and Q are *$\mathcal{F}$-conjugate*. Another consequence is that the endomorphisms of every object of $\mathcal{F}$ are automorphisms and we write $\mathrm{Aut}_{\mathcal{F}}(P)$ for $\mathrm{Hom}_{\mathcal{F}}(P, P)$.

Fusion systems form a convenient framework to study the p-local structure of finite groups. Let G be a finite group. For subgroup $H, K \leqslant G$ denote

$$N_G(H, K) = \{g \in G\colon\ gHg^{-1} \leqslant K\}.$$

Every element $g \in N_G(H, K)$ gives rise to a group monomorphism $c_g : H \to K$ where $c_g(h) = ghg^{-1}$. That is, c_g is a restriction of the inner automorphism c_g of G to H and K.

A Sylow p-subgroup S of G gives rise to a fusion system $\mathcal{F}_S(G)$ over S. Its objects are the subgroups of S. The morphisms $P \to Q$ in $\mathcal{F}_S(G)$ for $P, Q \leqslant S$ are the group monomorphisms $c_g : P \to Q$ for all $g \in N_G(P, Q)$. That is, the set of morphisms $P \to Q$ in $\mathcal{F}_S(G)$ is $N_G(P, Q)/C_G(P)$. The fusion system $\mathcal{F}_S(G)$ satisfies several crucial axioms which lead L. Puig to consider the class of saturated fusion systems.

2.1. Definition. (See [2].) Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup $P \leqslant S$ is called *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geqslant |C_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P. It is called *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geqslant |N_S(P')|$ for all P' which is $\mathcal{F}$-conjugate to P.

The fusion system $\mathcal{F}$ is called *saturated* if the following two conditions hold:

(I) If $P \leqslant S$ is fully normalized then it is fully centralized and $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.
(II) For every $P \leqslant S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ set

$$N_\varphi = \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\}.$$

If $\varphi(P)$ is fully centralized then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

The fusion system $\mathcal{F} = \mathcal{F}_S(G)$ associated to a finite group G is saturated by [2, Proposition 1.3].

2.2. Definition. Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup $P \leqslant S$ is $\mathcal{F}$*-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers.

Note that P is $\mathcal{F}$-centric if and only if $C_S(P') = Z(P')$ for any $P' \leqslant S$ which is $\mathcal{F}$-conjugate to P. In particular if P is $\mathcal{F}$-centric then all its $\mathcal{F}$-conjugates are fully centralized in $\mathcal{F}$. In addition any subgroup of S which contains P must also be $\mathcal{F}$-centric.

The collection of the $\mathcal{F}_S(G)$-centric subgroups has another description.

2.3. Definition. A p-subgroup $P \leqslant G$ is *p-centric* if $Z(P)$ is a Sylow p-subgroup of $C_G(P)$. Equivalently, $C_G(P) = Z(P) \times C'_G(P)$ where $C'_G(P)$ is a subgroup of $C_G(P)$ of order prime to p which is generated by the elements of $C_G(P)$ of order prime to p.

In particular $C'_G(P)$ is characteristic in $C_G(P)$. It is easy to see that $P \leqslant S$ is $\mathcal{F}_S(G)$-centric if and only if it is p-centric in G.

The following result will be useful later.

2.4. Proposition. *Let $\mathcal{F}$ be a saturated fusion system over S and let $P, Q \leqslant S$ be $\mathcal{F}$-centric subgroups. Consider a morphism $\varphi : Q \to P$ in $\mathcal{F}$ and an element $s \in N_S(Q)$ such that $c_x \circ \varphi = \varphi \circ c_s$ for some $x \in P$. Then there exists a subgroup $Q' \leqslant S$ and a morphism $\varphi' : Q' \to P$ in $\mathcal{F}$ such that $Q \leqslant Q'$ and $s \in Q'$ and $\varphi'|_Q = \varphi$.*

Proof. Set $Q' = \langle Q, s\rangle$. By [9, Theorem 3.8] there exists a morphism $\psi \in \mathcal{F}(Q', S)$ which extends $\varphi : Q \to S$. Now, $\psi \circ c_s \circ \psi^{-1} = c_{\psi(s)}$ as elements in $\mathrm{Aut}_{\mathcal{F}}(\psi(Q'))$ and by restriction to $\varphi(Q)$ we see that

$$c_{\psi(s)}|_{\varphi(Q)} = \psi \circ c_s \circ \psi^{-1}|_{\varphi(Q)} = \varphi \circ c_s \circ \varphi^{-1} = c_x \quad \big(\text{in } \mathrm{Aut}_{\mathcal{F}}\big(\varphi(Q)\big)\big).$$

Since $\varphi(Q)$ is $\mathcal{F}$-centric, $x^{-1}\psi(s) \in C_S(\varphi(Q)) \leqslant \varphi(Q) \leqslant P$, whence $\psi(s) \in P$. This shows that ψ restricts to a morphism $\varphi' \in \mathcal{F}(Q', P)$ which extends φ. □

2.5. Definition. The orbit category $\mathcal{O}(\mathcal{F})$ of a fusion system $\mathcal{F}$ over S has the same object set as $\mathcal{F}$ and the set of morphisms $P \to Q$ is the set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ modulu the action of $\mathrm{Inn}(Q)$ by postcomposition. The category $\mathcal{F}^c$ is the full subcategory of $\mathcal{F}$ on the set of the $\mathcal{F}$-centric subgroups. The category $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}(\mathcal{F})$ on the object set of $\mathcal{F}^c$.

The morphisms $P \to Q$ in $\mathcal{O}(\mathcal{F})$ will be denoted by $[\varphi]$ for some $\varphi : P \to Q$ in $\mathcal{F}$. Thus, $[\varphi] = [\psi]$ in $\mathcal{O}(\mathcal{F})$ if and only if there exists some $x \in Q$ such that $\psi = c_x \circ \varphi$ as morphisms in $\mathcal{F}$. It is easy to see that every endomorphism of P in $\mathcal{O}(\mathcal{F})$ is an isomorphism and we write $\mathrm{Out}_{\mathcal{F}}(P)$ for the automorphism group of P in $\mathcal{O}(\mathcal{F})$.

2.6. Proposition. *(See Puig [9, Corollary 3.6].) Let $\mathcal{F}$ be a saturated fusion over S. Then every morphism in $\mathcal{O}(\mathcal{F}^c)$ is an epimorphism (in the category-theory sense).*

Fix a saturated fusion system $\mathcal{F}$ over S and let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$. With the notation in 1.5 we observe $\mathrm{Out}_{\mathcal{F}}(K) = \mathcal{C}(K, K) = \mathrm{Aut}_{\mathcal{C}}(K)$ acts on $\mathcal{C}(K, P)$ for any $\mathcal{F}$-centric subgroups $K, P \leqslant S$. In particular, $\mathrm{Out}_S(K)$ whose elements are denoted $[c_s]$ for $s \in N_S(K)$, acts on $\mathcal{C}(K, P)$ and the set of orbits is denoted $\mathcal{C}(K, P)/\mathrm{Out}_S(K)$. The fixed point set of $[\alpha] \in \mathrm{Out}_{\mathcal{F}}(K)$ is denoted as usual by $\mathcal{C}(K, P)^{[\alpha]}$.

2.7. Proposition. *Let $\mathcal{F}$ be a saturated fusion system over S and let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$. Consider $\mathcal{F}$-centric subgroups $K, P \leqslant S$ and let H be the subgroup of S which is generated by K and some $s \in N_S(K)$. Then there is a bijection $\mathcal{C}(H, P) \approx \mathcal{C}(K, P)^{[c_s]}$ which is induced by the assignment $[\varphi] \mapsto [\varphi|_K]$.*

Proof. If $[\varphi] \in \mathcal{C}(H, P)$ then

$$[\varphi|_K] \circ [c_s] = [\varphi \circ c_s|_K] = [c_{\varphi(s)} \circ \varphi|_K] = [\varphi|_K]$$

because s normalizes K and $\varphi(s) \in P$. This shows that restriction $[\varphi] \mapsto [\varphi|_K]$ induces a well defined map $r : \mathcal{C}(H, P) \to \mathcal{C}(K, P)^{[c_s]}$. It is injective because by Corollary 2.6 the inclusion $K \leqslant H$ is an epimorphism in $\mathcal{C}$.

If $[\varphi] \in \mathcal{C}(K, P)^{[c_s]}$ then there exists some $x \in P$ such that $c_x \circ \varphi = \varphi \circ c_s$. Proposition 2.4 implies that φ extends to a morphism $\varphi' : H \to P$ and in particular $r([\varphi']) = [\varphi]$. This shows that r is also surjective. □

As a corollary we obtain the next result, see also [9, 4.3.2].

2.8. Proposition. *Set $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)$. Then $|\mathcal{C}(P, P')| = |\mathrm{Out}_{\mathcal{F}}(P')| \bmod p$ for any $P \leqslant P'$ in $\mathcal{F}^c$. In particular, $|\mathcal{C}(P, S)| = |\mathrm{Out}_{\mathcal{F}}(S)| \bmod p$ for any $P \in \mathcal{F}^c$.*

Proof. Use induction on $n = |P' : P|$, the case $n = 1$ being trivial. Choose $Q \leqslant P'$ which contains P and $|Q : P| = p$. By Proposition 2.7, $\mathcal{C}(Q, P') \approx \mathcal{C}(P, P')^{Q/P}$ and since $Q/P \cong \mathbb{Z}/p$ these sets have the same number of elements modulo p. By induction hypothesis $|\mathcal{C}(Q, P')| = |\mathrm{Out}_{\mathcal{F}}(P')| \bmod p$ and the result follows. □

As before let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$. The additive extension of $\mathcal{C}$, denoted $\mathcal{C}_\sqcup$ is defined as follows. Let $\hat{\mathcal{C}}$ denote the category of contravariant functors $\mathcal{C} \to \mathbf{Sets}$. Then $\mathcal{C}$ embeds as a full subcategory of $\hat{\mathcal{C}}$ via the Yoneda embedding $P \mapsto \mathcal{C}(-, P)$. Then $\mathcal{C}_\sqcup$ is the full subcategory of $\hat{\mathcal{C}}$ consisting of the functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ which are isomorphic to $\coprod_{i=1}^n \mathcal{C}(-, P_i)$ for some $n \geqslant 0$ and some $P_1, \dots, P_n \in \mathcal{C}$. We will write $P_1 \sqcup \cdots \sqcup P_n$ for this functor. By construction $\mathcal{C}_\sqcup$ is equipped with coproducts of finitely many objects. In fact $\mathcal{C}_\sqcup$ contains $\mathcal{C}$ as a full subcategory and every object of $\mathcal{C}_\sqcup$ is isomorphic to the coproduct of finitely many objects in $\mathcal{C}$. Moreover, for any $X, Y, Y' \in \mathcal{C}_\sqcup$,

$$\mathcal{C}_\sqcup(X, Y \sqcup Y') = \mathcal{C}_\sqcup(X, Y) \sqcup \mathcal{C}_\sqcup(X, Y').$$

Compare this with [8]. One observes that if for every $P, Q \in \mathcal{C}$ there are objects $A_1, \dots, A_n$ such that $\mathcal{C}(-, P) \times \mathcal{C}(-, Q) \cong \coprod_i \mathcal{C}(-, A_i)$ then $\mathcal{C}_\sqcup$ has products $\times_{\mathcal{C}}$ which distributes over the coproduct, namely

$$\left(\coprod_i P_i\right) \times_{\mathcal{C}} \left(\coprod_j Q_j\right) = \coprod_{i,j} P_i \times_{\mathcal{C}} Q_j.$$

This is the content of Puig's result in [9, Proposition 4.7]. Together with [9, Remark 4.6] we obtain the proposition below. See also the remark following it.

2.9. Proposition. *Let $\mathcal{F}$ be a saturated fusion on S and let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$. Then $\mathcal{C}_\sqcup$ admits distributive product $\times_{\mathcal{C}}$.*

2.10. Remark. By definition of $\mathcal{C}_\sqcup$, if P, Q are $\mathcal{F}$-centric subgroups, then $[P] \times_{\mathcal{C}} [Q] = \coprod_i [A_i]$ for some $\mathcal{F}$-centric subgroups A_i. The A_i's are described as follows.

Given $P, Q \leqslant S$ as above, let $K_{P,Q}$ denote the set of all the morphisms $[\alpha]: A \to Q$ in $\mathcal{C}$ where $A \leqslant P$ is $\mathcal{F}$-centric. We say that $[\gamma]: C \to Q$ extends $[\alpha]$ if $A \leqslant C \leqslant P$ and $[\alpha] = [\gamma|_A]$. We write $[\alpha] \preccurlyeq [\gamma]$. Then set $K_{P,Q}$ is partially ordered by the relation $\preccurlyeq$ of extension. The set of maximal elements of $K_{P,Q}$ under this relation is denoted $K_{P,Q}^{\max}$. Fix $[\alpha]: A \to Q$ in $K_{P,Q}$ and an element $x \in P$. Clearly $A^x = x^{-1}Ax$ is an $\mathcal{F}$-centric subgroup of P and we define an element $[\alpha] \cdot x$ in $K_{P,Q}$ by

$$[\alpha] \cdot x = [\alpha \circ c_x], \quad \text{where } c_x : A^x \to A \text{ is conjugation.}$$

There results an action of P on $K_{P,Q}$ which is easily seen to be order preserving. In particular P acts on the finite set $K_{P,Q}^{\max}$. Any choice of representatives $[\alpha_i]: A_i \to Q$ for the orbits $K_{P,Q}^{\max}/P$ gives the subgroups A_i. Moreover, $A_i \xrightarrow{\text{incl}} P$ and $A_i \xrightarrow{[\alpha_i]} Q$ give the structure maps $P \times_{\mathcal{C}} Q \to P$ and $P \times_{\mathcal{C}} Q \to Q$.

Note that the set of isomorphism classes of the objects of $\mathcal{C}_\sqcup$ form an abelian monoid with respect to the coproduct.

2.11. Definition. The Burnside ring $\mathcal{A}(\mathcal{F})$ of a saturated fusion system $\mathcal{F}$ on S is the group completion of the monoid of the isomorphism classes of the objects of $\mathcal{C}_\sqcup = \mathcal{O}(\mathcal{F}^c)_\sqcup$. The product in the ring is induced from the product $\times_{\mathcal{C}}$ in $\mathcal{C}_\sqcup$.

It is clear that the underlying abelian group of $\mathcal{A}(\mathcal{F})$ is free with one generator for every $\mathcal{F}$-conjugacy class of an $\mathcal{F}$-centric subgroup $P \leqslant S$ which we denote by $[P]$. The product on basis elements $[P]$ and $[Q]$ is given by $[P] \cdot [Q] = [P \times_{\mathcal{C}} Q]$.

3. Properties of the Burnside ring

We shall now fix a saturated fusion system $\mathcal{F}$ over S and let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$; See Definition 2.5. We shall write $[\mathcal{C}]$ for the set of the isomorphism classes of the objects of $\mathcal{C}$, that is, $[\mathcal{C}]$ is the set of the $\mathcal{F}$-conjugacy classes of the $\mathcal{F}$-centric subgroups of S. The elements of $[\mathcal{C}]$ are denoted $[P]$ for an $\mathcal{F}$-centric $P \leqslant S$. Obviously, $[\mathcal{C}]$ is a finite set.

We now consider the ring $\prod_{[\mathcal{C}]} \mathbb{Z}$. As an abelian group it is free with the set $[\mathcal{C}]$ as a natural choice of a basis. Thus, every element in $\prod_{[\mathcal{C}]} \mathbb{Z}$ has the form $\sum_{[Q] \in [\mathcal{C}]} n_Q \cdot [Q]$ and we shall sometimes abbreviate by writing (n_Q) for this element. The product in this ring is defined coordinate-wise, namely $(n_Q) \cdot (m_Q) = (n_Q m_Q)$.

As an abelian group $\mathcal{A}(\mathcal{F}) = \bigoplus_{[\mathcal{C}]} \mathbb{Z}$ and we let $[\mathcal{C}]$ be a basis. By determining its values on basis elements, we obtain a homomorphism of groups

$$\Phi : \mathcal{A}(\mathcal{F}) \to \prod_{[H]\in[\mathcal{C}]} \mathbb{Z}, \qquad [P] \mapsto \sum_{[H]\in[\mathcal{C}]} \left|\mathcal{C}(H, P)\right| \cdot [H]. \tag{3.1}$$

In fact, this is a ring homomorphism because for every $P, Q \in \mathcal{C}$ and every $H \in \mathcal{C}$ we have $|\mathcal{C}(H, P \times_{\mathcal{C}} Q)| = |\mathcal{C}(H, P)| \cdot |\mathcal{C}(H, Q)|$.

Thus, the homomorphism Φ is represented by a matrix $\mathbf{m}$ whose entries are

$$\mathbf{m}\big([Q], [P]\big) = \left|\mathcal{C}(Q, P)\right|, \quad [Q], [P] \in [\mathcal{C}].$$

In the language of tom-Dieck, this is the analogue of the "table of marks" for the Burnside ring of a finite group.

It is clearly possible to totally order the set $[\mathcal{C}]$ in such a way that $[H] \preccurlyeq [K]$ implies $|H| \leqslant |K|$. Using this total order the matrix $\mathbf{m}$ becomes upper triangular and its diagonal entries are $|\mathrm{Out}_{\mathcal{F}}(Q)|$ for $[Q] \in [\mathcal{C}]$.

The rational Burnside ring

We shall use the symbol $Q \simeq_{\mathcal{F}} P$ for the statement that Q and P are $\mathcal{F}$-conjugate subgroups in a fusion system $\mathcal{F}$.

3.2. Proposition. *Let $\mathcal{F}$ be a saturated fusion system over S and let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$. Then, for every $\mathcal{F}$-centric subgroups $Q, P \leqslant S$*

$$\left|\mathcal{C}(Q, P)\right| = \frac{|Z(Q)| \cdot |\mathrm{Aut}_{\mathcal{F}}(Q)|}{|P|} \cdot \left|\{T \leqslant P \colon T \simeq_{\mathcal{F}} Q\}\right|.$$

Proof. Consider the action of P on $\mathcal{F}(Q, P)$ by conjugation. The stabilizer group P_φ of $\varphi \in \mathcal{F}(Q, P)$ is $C_P(\varphi(Q)) = Z(\varphi(Q)) = \varphi(Z(Q))$ because Q is $\mathcal{F}$-centric. In particular $|P_\varphi| = |Z(Q)|$ for all φ. Now, $\mathcal{C}(Q, P)$ is the set of orbits of P in this action, so by the "orbit-stabilizer property"

$$\left|\mathcal{C}(Q, P)\right| = \frac{1}{|P|} \cdot \sum_{\varphi\in\mathcal{F}(Q,P)} |P_\varphi| = \frac{|Z(Q)|}{|P|} \cdot \left|\mathrm{Hom}_{\mathcal{F}}(Q, P)\right|. \tag{1}$$

The assignment $\varphi \mapsto \varphi(Q)$ defines a surjective function

$$\mathcal{F}(Q, P) \to \{T \leqslant P \colon T \simeq_{\mathcal{F}} Q\}.$$

The fiber of this function over an element T is clearly $\mathrm{Iso}_{\mathcal{F}}(Q, T)$ which is, in turn, equipotent to $\mathrm{Aut}_{\mathcal{F}}(Q)$. Therefore $|\mathcal{F}(Q, P)| = |\mathrm{Aut}_{\mathcal{F}}(Q)| \cdot |\{T \leqslant P \colon T \simeq_{\mathcal{F}} Q\}|$. Combining this with (1) yields the result. $\square$

Let $\mathcal{P}$ be a finite poset. The Möbius function of $\mathcal{P}$ is a function $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ which is defined recursively by the requirement that $\mu(a,b) = 0$ unless $a \leqslant b$ and the following equivalent equalities hold for any $a \leqslant b$,

$$\sum_{a \leqslant c \leqslant b} \mu(a,c) = \delta_{a,b} \quad \text{and} \quad \sum_{a \leqslant c \leqslant b} \mu(c,b) = \delta_{a,b}$$

where δ is the Kronecker delta function. (See e.g. Solomon [12] or Rota [10].)

The set of $\mathcal{F}$-centric subgroups of S forms a poset and we let $\mu_{\mathcal{F}}$ denote its Möbius function. Using the homomorphism (3.1) we are now ready to describe the rational Burnside ring of $\mathcal{F}$, cf. e.g. Solomon [12], Gluck [7] or [14].

3.3. Theorem. *Let $\mathcal{F}$ be a saturated fusion system over S and let $[\mathcal{C}]$ denote the set of the $\mathcal{F}$-conjugacy classes of the $\mathcal{F}$-centric subgroups of S. Then $\mathbb{Q} \otimes \Phi$ is an isomorphism of rings $\mathbb{Q} \otimes \mathcal{A}(\mathcal{F}) \approx \prod_{[\mathcal{C}]} \mathbb{Q}$. In particular $\mathbb{Q} \otimes \mathcal{A}(\mathcal{F})$ is a semisimple algebra with one primitive idempotent e_P for every element $[P]$ of $[\mathcal{C}]$. In fact, using $[\mathcal{C}]$ as a basis for $\mathbb{Q} \otimes \mathcal{A}(\mathcal{F})$,*

$$e_P = \frac{1}{|P| \cdot |\mathrm{Out}_{\mathcal{F}}(P)|} \cdot \sum_{Q \leqslant P,\, Q \in \mathcal{F}^c} \bigl(|Q| \cdot \mu_{\mathcal{F}}(Q,P)\bigr) \cdot [Q].$$

The summation is over $Q \leqslant P$ such that $Q \in \mathcal{F}^c$.

Proof. Consider the ring homomorphism Φ defined in (3.1). We have already remarked that by appropriately ordering the elements of $[\mathcal{C}]$, the matrix $\mathbf{m}$ which represents Φ becomes an upper triangular with non-zero values on the diagonal. Therefore, $\mathbb{Q} \otimes \Phi$ is an isomorphism. In particular $\mathbb{Q} \otimes \mathcal{A}(\mathcal{F})$ is semisimple with primitive idempotents $(\mathbb{Q} \otimes \Phi)^{-1}([P])$ for every basis element $[P]$ of $\prod_{[\mathcal{C}]} \mathbb{Q}$.

To avoid clutter we shall write Φ for $\mathbb{Q} \otimes \Phi$ and for every $[Q] \in [\mathcal{C}]$ we write Φ_Q for the projection of Φ onto the factor of $[Q]$ in $\prod_{[\mathcal{C}]} \mathbb{Q}$. It remains to show that $\Phi(e_P) = [P]$ for all $[P]$. For every $[Q] \in [\mathcal{C}]$ use Proposition 3.2 and the definition of Φ to deduce that

$$\begin{aligned}
&\Phi_Q\biggl(\sum_{H \leqslant P, H \in \mathcal{F}^c} |H| \cdot \mu_{\mathcal{F}}(H,P) \cdot [H]\biggr) \\
&\quad = \sum_{H \leqslant P, H \in \mathcal{F}^c} |H| \cdot \mu_{\mathcal{F}}(H,P) \cdot \bigl|\mathcal{C}(Q,H)\bigr| \\
&\quad = \sum_{H \leqslant P, H \in \mathcal{F}^c} \bigl(|H| \cdot \mu_{\mathcal{F}}(H,P)\bigr) \cdot \frac{|Z(Q)| \cdot |\mathrm{Aut}_{\mathcal{F}}(Q)|}{|H|} \cdot \bigl|\{T \leqslant H \colon T \simeq_{\mathcal{F}} Q\}\bigr| \\
&\quad = \bigl|Z(Q)\bigr| \cdot \bigl|\mathrm{Aut}_{\mathcal{F}}(Q)\bigr| \cdot \sum_{H \leqslant P, H \in \mathcal{F}^c} \; \sum_{T \leqslant H, T \simeq_{\mathcal{F}} Q} \mu_{\mathcal{F}}(H,P) \\
&\quad = \bigl|Z(Q)\bigr| \cdot \bigl|\mathrm{Aut}_{\mathcal{F}}(Q)\bigr| \cdot \sum_{T \leqslant P, T \simeq_{\mathcal{F}} Q} \; \sum_{H \leqslant P, T \leqslant H} \mu_{\mathcal{F}}(H,P).
\end{aligned} \tag{1}$$

Here we used the fact that if $T \simeq_{\mathcal{F}} Q$ then T is $\mathcal{F}$-centric, hence so is every subgroup $H \leqslant S$ containing T. By the recursive relation of $\mu_{\mathcal{F}}$ and Proposition 3.2 we see that (1) is equal to

$$\big|Z(Q)\big| \cdot \big|\mathrm{Aut}_{\mathcal{F}}(Q)\big| \cdot \sum_{T \leqslant P, T \simeq_{\mathcal{F}} Q} \delta_{T,P} = \begin{cases} 0 & \text{if } Q \not\simeq_{\mathcal{F}} P, \\ |P| \cdot |\mathrm{Out}_{\mathcal{F}}(P)| & \text{if } Q \simeq_{\mathcal{F}} P. \end{cases}$$

Therefore, $\Phi_Q(e_P) = \delta_{[Q],[P]}$, i.e. $\Phi(e_P) = [P]$. □

The p-local Burnside ring

We shall now study $\mathcal{A}(\mathcal{F})_{(p)} = \mathbb{Z}_{(p)} \otimes \mathcal{A}(\mathcal{F})$ and prove that it has a unit.

We denote $\mathcal{O}(\mathcal{F}^c)$ by $\mathcal{C}$ and let $[\mathcal{C}]$ denote the set of the isomorphism classes of its objects. Consider Φ from (3.1) and denote $\Phi_{(p)} = \mathbb{Z}_{(p)} \otimes \Phi$. Clearly, its domain $\mathcal{A}(\mathcal{F})_{(p)}$ and codomain $\prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$ are free $\mathbb{Z}_{(p)}$-modules with basis $[\mathcal{C}]$.

3.4. Theorem. *Let $\mathcal{F}$ be a saturated fusion system over S. For $\mathcal{F}$-centric subgroups $H, K \leqslant S$ such that K is fully normalized in $\mathcal{F}$ set*

$$n(K, H) = \big|\big\{[c_s] \in \mathrm{Out}_S(K)\colon \langle s, K\rangle \simeq_{\mathcal{F}} H\big\}\big|.$$

Then an element $(y_H) \in \prod_{[H] \in [\mathcal{C}]} \mathbb{Z}_{(p)}$ is in the image of $\Phi_{(p)}$ if and only if the following congruences hold for any fully $\mathcal{F}$-normalized $\mathcal{F}$-centric subgroup $K \leqslant S$

$$\sum_{[H] \in [\mathcal{C}]} n(K, H) \cdot y_H \equiv 0 \mod \big(\big|\mathrm{Out}_S(K)\big|\big) \quad \text{in } \mathbb{Z}_{(p)}. \tag{1}$$

Proof. First, note that $n(K, H)$ is well defined because if $s, s' \in N_S(K)$ define the same element in $\mathrm{Out}_S(K)$ then $s^{-1}s' \in K$ because K is $\mathcal{F}$-centric so $C_S(K) \leqslant K$. It follows that $\langle s, K\rangle = \langle s', K\rangle$. It is also clear that $n(K, H) = n(K, H')$ if $H \simeq_{\mathcal{F}} H'$.

We shall now fix once and for all representatives H for the elements $[H] \in [\mathcal{C}]$ which are fully normalized in $\mathcal{F}$. We also totally order $[\mathcal{C}]$ in such a way that $[H] \preccurlyeq [K]$ implies that $|H| \leqslant |K|$.

With respect to the ordered basis $[\mathcal{C}]$ of $\mathcal{A}(\mathcal{F})_{(p)}$ and $\prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$, the homomorphism Φ is represented by an upper triangular matrix $\mathbf{m}$ whose entries are

$$\mathbf{m}\big([H], [P]\big) = \big|\mathcal{C}(H, P)\big|.$$

Using the choice of representatives H for the elements of $[\mathcal{C}]$ we now define matrices $\mathbf{n}$ and $\mathbf{t}$ with the same dimensions as $\mathbf{m}$ and with the set $[\mathcal{C}]$ as basis whose entries are

$$\mathbf{n}\big([K], [H]\big) = n(K, H), \quad [K], [H] \in [\mathcal{C}] \quad \text{and}$$
$$\mathbf{t}\big([K], [P]\big) = \big|\mathcal{C}(K, P)/\mathrm{Out}_S(K)\big|, \quad [K], [P] \in [\mathcal{C}].$$

In addition let $\mathbf{d}$ be the diagonal matrix whose diagonal entries are

$$\mathbf{d}\big([K], [K]\big) = \big|\mathrm{Out}_S(K)\big|, \quad [K] \in [\mathcal{C}].$$

We now note that if K and K' are $\mathcal{F}$-conjugate subgroups of S which are fully normalized in $\mathcal{F}$, then axiom (I) of saturation (Definition 2.1) and Sylow's theorems imply that there is an isomorphism $\psi : K \to K'$ such that $\psi\,\mathrm{Out}_S(K)\psi^{-1} = \mathrm{Out}_S(K')$. Therefore $N_\psi = N_S(K)$ and

ψ extends to $\tilde{\psi}: N_S(K) \to N_S(K')$. This shows that the definition of $\mathbf{n}, \mathbf{t}$ and $\mathbf{d}$ is independent of our choice of the fully $\mathcal{F}$-normalized representatives K for the elements $[K]$ of $[\mathcal{C}]$.

Claim 1. *The matrices* $\mathbf{n}$ *and* $\mathbf{t}$ *are invertible over* $\mathbb{Z}_{(p)}$.

Proof. The choice of the total order of $[\mathcal{C}]$ implies $n(K, H) = 0$ if $[H] \gneqq [K]$ because in this case either $|H| < |K|$ or H and K are not $\mathcal{F}$-conjugate. Therefore $\mathbf{n}$ is an upper triangular matrix. Also $n(K, K) = 1$ because $\langle K, s\rangle = K$ if and only if $s \in K$. Hence the diagonal entries of $\mathbf{n}$ are equal to 1 and therefore $\mathbf{n}$ is invertible.

Similarly, $\mathbf{t}([K], [P]) = 0$ if $[P] < [K]$ so $\mathbf{t}$ is upper triangular. Its diagonal entries are equal to

$$\left|\mathrm{Out}_{\mathcal{F}}(K) : \mathrm{Out}_S(K)\right| \neq 0 \mod p$$

because the representative K of $[K] \in [\mathcal{C}]$ is fully normalized in $\mathcal{F}$. They are therefore units in $\mathbb{Z}_{(p)}$, hence $\mathbf{t}$ is invertible. □

Claim 2. $\mathbf{n} \cdot \mathbf{m} = \mathbf{d} \cdot \mathbf{t}$.

Proof. Fix some $[K], [P]$ in $[\mathcal{C}]$ and recall that $\mathrm{Out}_S(K)$ acts on $\mathcal{C}(K, P)$. Since every subgroup of S which contains K is $\mathcal{F}$-centric, the $([K], [P])$-entry of $\mathbf{n} \cdot \mathbf{m}$ is

$$\begin{aligned}
&\sum_{[H]\in[\mathcal{C}]} \mathbf{n}\big([K], [H]\big) \cdot \mathbf{m}\big([H], [P]\big) \\
&\quad = \sum_{[H]\in[\mathcal{C}]} \left|\left\{[c_s] \in \mathrm{Out}_S(K)\colon\ \langle K, s\rangle \simeq_{\mathcal{F}} H\right\}\right| \cdot \left|\mathcal{C}(H, P)\right| \\
&\quad = \sum_{[H]\in[\mathcal{C}]} \left(\sum_{[c_s]\in \mathrm{Out}_S(K),\, \langle s, K\rangle \simeq_{\mathcal{F}} H} \left|\mathcal{C}(H, P)\right| \right) \\
&\quad = \sum_{[c_s]\in \mathrm{Out}_S(K)} \left|\mathcal{C}\big(\langle s, K\rangle, P\big)\right| \quad \text{(by Proposition 2.7)} \\
&\quad = \sum_{[c_s]\in \mathrm{Out}_S(K)} \left|\mathcal{C}(K, P)^{[c_s]}\right| \quad \text{(by Frobenius's Lemma)} \\
&\quad = \left|\mathrm{Out}_S(K)\right| \cdot \left|\mathcal{C}(K, P)/\mathrm{Out}_S(K)\right| = \left|\mathrm{Out}_S(K)\right| \cdot \mathbf{t}\big([K], [P]\big)
\end{aligned}$$

which is the $([K], [P])$-entry of $\mathbf{d} \cdot \mathbf{t}$. □

We now prove that every element in the image of $\Phi_{(p)}$ satisfies the congruences (1). By linearity it suffices to prove this for elements of the form

$$\Phi\big([P]\big) = \sum_{[H]\in[\mathcal{C}]} \left|\mathcal{C}(H, P)\right| \cdot [H] = \sum_{[H]\in[\mathcal{C}]} \mathbf{m}\big([H], [P]\big) \cdot [H]$$

which we now denote by $(y_H) \in \prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$, that is $y_H = |\mathcal{C}(H, P)|$.

For every $K \leqslant S$ which is fully normalized in $\mathcal{F}$ we have seen that it may be assumed to be the representative of $[K]$ in the definitions of **m**, **n**, **t** and **d** and therefore Claim 2 implies

$$\sum_{[H]\in[\mathcal{C}]} n(K,H)\cdot y_H = \sum_{[H]\in[\mathcal{C}]} \mathbf{n}\big([K],[H]\big)\cdot \mathbf{m}\big([H],[P]\big)$$
$$= \big|\mathrm{Out}_S(K)\big|\cdot \mathbf{t}\big([K],[P]\big) = 0 \mod \big(\big|\mathrm{Out}_S(K)\big|\big).$$

That is, $(y_H) = \Phi_{(p)}([P])$ satisfies the congruence (1).

Conversely, assume that $(y_H) \in \prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$ satisfies all the congruences (1). We view (y_H) as a column vector and note that satisfying these congruences is equivalent to the existence of a column vector $(z_H) \in \prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$ such that

$$\mathbf{n}\cdot (y_H) = \mathbf{d}\cdot (z_H).$$

Since **n** and **t** are invertible by Claim 1, we deduce from Claim 2 that

$$(y_H) = \mathbf{n}^{-1}\cdot \mathbf{d}\cdot (z_H) = \mathbf{m}\cdot \mathbf{t}^{-1}\cdot (z_H) \in \mathrm{Im}\,\mathbf{m} = \mathrm{Im}\,\Phi_{(p)}$$

because the matrix **m** represents $\Phi_{(p)}$. This completes the proof. $\square$

3.5. Corollary. *Let $\mathcal{F}$ be a saturated fusion system over S. Then $\mathcal{A}(\mathcal{F})_{(p)}$ is a commutative ring with a unit. More precisely, let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$ and $[\mathcal{C}]$ the set of isomorphism classes of its objects. Then $\Phi_{(p)}$ embeds $\mathcal{A}(\mathcal{F})_{(p)}$ as a subring of $\prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$ and moreover, the cokernel of $\Phi_{(p)}$ is a finite abelian p-group. Furthermore the unit of $\prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$ is contained in $\mathcal{A}(\mathcal{F})_{(p)}$.*

Proof. First, $\ker \Phi_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module because it is a submodule of the free $\mathbb{Z}_{(p)}$-module $\mathcal{A}(\mathcal{F})_{(p)}$ (note that $\mathbb{Z}_{(p)}$ is a principal ideal domain). Similarly $\operatorname{coker} \Phi_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module. Since $\mathbb{Q}\otimes -$ is an exact functor, Theorem 3.3 implies that $\ker \Phi_{(p)} = 0$ and that $\operatorname{coker} \Phi_{(p)}$ must be a finite abelian p-group. In particular $\mathcal{A}(\mathcal{F})_{(p)}$ is a commutative ring because $\prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$ is commutative.

Now we apply Theorem 3.4 to show that the unit $(1)_H$ of $\prod_{[\mathcal{C}]} \mathbb{Z}_{(p)}$ is in the image of $\Phi_{(p)}$. For every $Q \in \mathcal{F}^c$ which is fully normalized we have

$$\sum_{P\in[\mathcal{C}]} n(Q,P)\cdot 1 = \sum_{P\in[\mathcal{C}]}\Bigg(\sum_{[c_s]\in N_S(Q)/Q,\,\langle s,Q\rangle\simeq_{\mathcal{F}} P} 1\Bigg) = \sum_{[c_s]\in N_S(Q)/Q} 1$$
$$= \big|N_S(Q)/Q\big| \equiv 0 \mod \big(\big|\mathrm{Out}_S(Q)\big|\big),$$

because every subgroup of S which contains Q must be $\mathcal{F}$-centric. $\square$

3.6. Remark. Recall that for a finite group G the Burnside ring $\mathcal{A}(G)$ has as unit the unique G-set of cardinal 1. However, $\mathcal{A}(\mathcal{F})$ has no unit in general. To see this notice that the "table of marks" **m** defining the monomorphism (3.1) is an upper triangular matrix and that $\mathbf{m}([S],[S]) = |\mathrm{Out}_{\mathcal{F}}(S)|$ is the only non-zero entry in its row. Thus if $\mathrm{Out}_{\mathcal{F}}(S) \neq 1$ there are no (integral) idempotents in $\mathcal{A}(\mathcal{F})$.

The prime spectrum

We shall now study the set of the prime ideals of $\mathcal{A}(\mathcal{F})_{(p)}$. Here it is subsumed in the definition of prime ideal that a prime ideal is strictly included in the ring. The prime spectrum of the Burnside ring of a finite group was determined by Dress [6]. Here we describe the prime spectrum of the p-localized Burnside ring $\mathcal{A}(\mathcal{F})_{(p)}$ of a saturated fusion system $\mathcal{F}$.

Throughout we shall fix a saturated fusion system $\mathcal{F}$ over S and let $\mathcal{C}$ denote $\mathcal{O}(\mathcal{F}^c)$. The $\mathcal{F}$-conjugacy class of an object $H \in \mathcal{C}$ is denoted $[H]$ and we let $[\mathcal{C}]$ denote the collection of these classes. Clearly $\mathcal{C}$ is a poset under inclusion of groups and $[\mathcal{C}]$ is a poset as well.

3.7. Definition. Let $[H]$ be an $\mathcal{F}$-conjugacy class of some $H \in \mathcal{C}$ and let q denote either the integer p or 0. Define $\mathfrak{p}_{[H],q}$ as the kernel of the ring homomorphism

$$\mathcal{A}(\mathcal{F})_{(p)} \xrightarrow{\Phi_{(p)}} \prod_{[\mathcal{C}]} \mathbb{Z}_{(p)} \xrightarrow{\mathrm{proj}_{[H]}} \mathbb{Z}_{(p)} \twoheadrightarrow \mathbb{Z}_{(p)}/(q)$$

which we denote by $\pi_{[H],q}$.

We observe that the homomorphism in 3.7 must be surjective because $\mathcal{A}(\mathcal{F})_{(p)}$ is a unital ring by Corollary 3.5. Its image is therefore either $\mathbb{Z}_{(p)}$ or $\mathbb{F}_p$, whence $\mathfrak{p}_{[H],q}$ are prime ideals of $\mathcal{A}(\mathcal{F})_{(p)}$.

Our next result is that these are the only prime ideals of $\mathcal{A}(\mathcal{F})_{(p)}$. Recall that an additive basis for $\mathcal{A}(\mathcal{F})_{(p)}$ is the set $[\mathcal{C}]$.

3.8. Proposition. *Let $\mathfrak{p}$ be a prime ideal in $\mathcal{A} = \mathcal{A}(\mathcal{F})_{(p)}$ and let q be the characteristic of $\mathcal{A}/\mathfrak{p}$. Then*

(a) *among all the classes $[K] \in [\mathcal{C}]$ whose image under the projection $\mathcal{A} \to \mathcal{A}/\mathfrak{p}$ is non-zero, there exists a unique minimal class $[H]$.*
(b) *Either $q = 0$ or $q = p$ and moreover $\mathfrak{p} = \mathfrak{p}_{[H],q}$.*

Proof. (a) Let $\pi : \mathcal{A} \to \mathcal{A}/\mathfrak{p}$ denote the projection. Since $\mathcal{A}$ is generated by the classes $[K] \in [\mathcal{C}]$ and $\mathcal{A}/\mathfrak{p} \neq 0$ there must exist some $[K]$ such that $\pi([K]) \neq 0$. Choose some $[H]$ which is minimal in the poset $[\mathcal{C}]$ with this property. Given an arbitrary $[Q] \in [\mathcal{C}]$, we recall from Proposition 2.9 and Remark 2.10 that $[H] \times_{\mathcal{C}} [Q] \cong \coprod_i [A_i]$ where $A_i \leqslant H$. Since $\mathcal{C}([H], [H \times_{\mathcal{C}} Q]) = \mathcal{C}([H], [H]) \times \mathcal{C}([H], [Q])$ and since $\mathcal{C}([H], [A_i]) = \emptyset$ unless $A_i = H$, we see that

$$[H \times_{\mathcal{C}} Q] = \left|\mathcal{C}(H, Q)\right| \cdot [H] + \sum_i [A_i] \quad (A_i \lneq H).$$

From the minimality of H we now deduce that $\pi([H]) \cdot \pi([Q]) = |\mathcal{C}(H, Q)| \cdot \pi([H])$ and since $\mathcal{A}/\mathfrak{p}$ is an integral domain with a unit,

$$\pi\big([Q]\big) = \left|\mathcal{C}(H, Q)\right| \cdot 1_{\mathcal{A}/\mathfrak{p}}. \tag{1}$$

If $[Q] \in [\mathcal{C}]$ satisfies $\pi([Q]) \neq 0$ then $\mathcal{C}(H, Q) \neq \emptyset$, namely $[H] \preccurlyeq [Q]$ in $[\mathcal{C}]$. If, in addition, $[Q]$ is also minimal with respect to this property, then $[H] = [Q]$.

(b) Clearly $\mathcal{A}/\mathfrak{p}$ is a $\mathbb{Z}_{(p)}$-module and since $\mathcal{A}$ is generated by the classes $[Q]$, Eq. (1) implies that $\mathcal{A}/\mathfrak{p} = \mathbb{Z}_{(p)}/(q)$. Since $\mathcal{A}/\mathfrak{p} \neq 0$ then either $q = p$ or $q = 0$. It now follows by inspection of $\Phi_{(p)}$ (see (3.1)) that the homomorphism in Definition 3.7 coincides with $\pi : \mathcal{A} \to \mathcal{A}/\mathfrak{p} \cong \mathbb{Z}_{(p)}/(q)$. In particular it follows that $\mathfrak{p} = \ker(\pi) = \mathfrak{p}_{[H],q}$. $\square$

It remains to understand the relationship between the ideals $\mathfrak{p}_{[H],q}$.

3.9. Proposition. *The following holds in $\mathcal{A}(\mathcal{F})_{(p)}$.*

(a) $\mathfrak{p}_{[H],0} = \mathfrak{p}_{[K],0}$ *if and only if* $[H] = [K]$.
(b) $\mathfrak{p}_{[H],0} \subsetneq \mathfrak{p}_{[H],p}$ *for any* $[H] \in [\mathcal{C}]$.
(c) $\mathfrak{p}_{[H],p} = \mathfrak{p}_{[S],p}$ *for all* $[H] \in [\mathcal{C}]$ *where S is the Sylow of $\mathcal{F}$.*

Proof. (a) One implication is trivial. Assume that $\mathfrak{p}_{[H],0} = \mathfrak{p}_{[K],0}$. Observe that $[H] \notin \mathfrak{p}_{[H],0}$ because $\pi_{[H],0}([H]) = |\mathcal{C}(H,H)| \neq 0$ (see Definition 3.7). Therefore $[H] \notin \mathfrak{p}_{[K],0}$, namely $|\mathcal{C}(K,H)| = \pi_{[K],0}([H]) \neq 0$. Similarly $|\mathcal{C}(H,K)| \neq 0$ which implies that H and K are $\mathcal{F}$-conjugate.

(b) Clearly $\mathfrak{p}_{[H],0} \subseteq \mathfrak{p}_{[H],p}$. Now, $\pi_{[H],q}$ are surjective so $\mathcal{A}(\mathcal{F})_{(p)}/\mathfrak{p}_{[H],0} \cong \mathbb{Z}_{(p)}$ while $\mathcal{A}(\mathcal{F})_{(p)}/\mathfrak{p}_{[H],p} \cong \mathbb{F}_p$.

(c) Consider some $[Q] \notin \mathfrak{p}_{[H],p}$, that is $|\mathcal{C}(H,Q)| = \pi_{[H],p}([Q]) \neq 0 \bmod p$. In particular H is $\mathcal{F}$-conjugate to a subgroup of Q. By Proposition 2.8, $|\mathcal{C}(H,Q)| = |\mathrm{Out}_{\mathcal{F}}(Q)| \bmod (p)$ whence $|\mathrm{Out}_{\mathcal{F}}(Q)| \neq 0 \bmod (p)$. The axioms for saturated fusion system (Definition 2.1) imply that $Q = S$. That is, the only class $[Q]$ which projects non-trivially in $\mathcal{A}(\mathcal{F})_{(p)}/\mathfrak{p}_{[H],p}$ is $[S]$. Proposition 3.8 now implies that $\mathfrak{p}_{[H],p} = \mathfrak{p}_{[S],p}$. $\square$

3.10. Corollary. *Let $\mathcal{F}$ be a saturated fusion system over S and let $\mathcal{A}(\mathcal{F})_{(p)}$ be its p-localized Burnside ring. Then $\mathcal{A}(\mathcal{F})_{(p)}$ is a local ring with a maximal ideal $\mathfrak{m} = \mathfrak{p}_{[S],p}$. The remaining prime ideals in $\mathcal{A}(\mathcal{F})_{(p)}$ have the form $\mathfrak{p}_{[H],0}$; they are all distinct and none of them is contained in the other.*

Proof. Every prime ideal $\mathfrak{p}$ in $\mathcal{A} = \mathcal{A}(\mathcal{F})_{(p)}$ has the form $\mathfrak{p}_{[H],q}$ by Proposition 3.8. Parts (b) and (c) of Proposition 3.9 show that $\mathfrak{p}_{[S],p}$ is a unique maximal ideal in $\mathcal{A}$. The remaining prime ideals have the form $\mathfrak{p}_{[H],0}$ and they are all distinct by part (a). Suppose that $\mathfrak{p}_{[H],0} \subseteq \mathfrak{p}_{[K],0}$. There results a surjective ring homomorphism

$$\mathbb{Z}_{(p)} \cong \mathcal{A}/\mathfrak{p}_{[H],0} \to \mathcal{A}/\mathfrak{p}_{[K],0} \cong \mathbb{Z}_{(p)}$$

which must be an isomorphism, whence $\mathfrak{p}_{[H],0} = \mathfrak{p}_{[K],0}$. $\square$

3.1. Relationship with the classical Burnside ring

Let G be a finite group. The set of all its subgroups is denoted $S(G)$. The conjugacy class of $H \leqslant G$ is denoted $[H]$. The Burnside ring $\mathcal{A}(G)$ of G is isomorphic to the free abelian group $\bigoplus_{[S(G)]} \mathbb{Z}$ where product of basis elements $[H]$ and $[K]$ is given by the double coset formula $[H] \cdot [K] = \sum_{g \in K\backslash G/H} [K^g \cap H]$.

A *collection* in G is a subset $\mathcal{H}$ of $S(G)$ which is closed to conjugation in G. The set of the conjugacy classes of the elements of $\mathcal{H}$ is denoted $[\mathcal{H}]$. Let $\mathcal{A}(G;\mathcal{H})$ be the subgroup of $\mathcal{A}(G)$ generated by the basis elements $[H]\in[\mathcal{H}]$. Thus,

$$\mathcal{A}(G;\mathcal{H}) = \bigoplus_{[H]\in[\mathcal{H}]} \mathbb{Z} \leqslant \mathcal{A}(G).$$

The double coset formula implies that $\mathcal{A}(G;\mathcal{H})$ is an ideal in $\mathcal{A}(G)$ if $\mathcal{H}$ is closed to formation of subgroups. For example, the collection $S_p(G)$ of all the p-subgroups of G has this property and it defines an ideal

$$\mathcal{A}(G;p) = \mathcal{A}\big(G; S_p(G)\big) \lhd \mathcal{A}(G).$$

The collection $S_p(G)$ contains the collections $S_p^{\mathrm{cent}}(G)$ of all the p-centric subgroups and the collection $S_p^{\neg\mathrm{cent}}(G)$ of all the p-subgroups of G that are not p-centric; See Definition 2.3. The discussion after 2.3 shows that $S_p^{\neg\mathrm{cent}}(G)$ is closed to formation of subgroups and defines an ideal

$$\mathcal{A}(G; p\text{-}\neg\mathrm{cent}) \lhd \mathcal{A}(G)$$

which is clearly contained in $\mathcal{A}(G;p)$. There results a quotient ring

$$\mathcal{A}^{p\text{-cent}}(G) = \mathcal{A}(G;p)/\mathcal{A}(G; p\text{-}\neg\mathrm{cent}).$$

As an abelian group it is free with basis $[S_p^{\mathrm{cent}}(G)]$. The product of basis elements $[P]$ and $[Q]$ is $\sum_g [Q^g\cap P]$ where the sum ranges through the double cosets QgP such that $Q^g\cap P$ is p-centric.

We can tensor the constructions above with $\mathbb{Z}_{(p)}$. We denote $\mathbb{Z}_{(p)}\otimes\mathcal{A}(G)$ by $\mathcal{A}(G)_{(p)}$. Similarly we consider $\mathcal{A}(G;p)_{(p)}$ and $\mathcal{A}(G;p\text{-}\neg\mathrm{cent})_{(p)}$ and $\mathcal{A}^{p\text{-cent}}(G)_{(p)}$. We remark that the latter is the free $\mathbb{Z}_{(p)}$ module with basis $[S_p^{\mathrm{cent}}(G)]$ with the same formula for the product of basis elements and moreover

$$\mathcal{A}^{p\text{-cent}}(G)_{(p)} = \mathcal{A}(G;p)_{(p)}/\mathcal{A}(G; p\text{ -}\neg\mathrm{cent})_{(p)}.$$

3.11. Theorem. *Let $\mathcal{F}$ be the fusion system associated to a finite group G and a Sylow p-subgroup S. Then the rings $\mathcal{A}(\mathcal{F})_{(p)}$ and $\mathcal{A}^{p\text{-cent}}(G)_{(p)}$ are isomorphic.*

Proof. Given a G-set X we denote by X^H the points of X fixed by H. A subgroup $K\leqslant G$ gives rise to a transitive G-set G/K by left translations. The G-sets G/K and G/K' are isomorphic if and only if K and K' are conjugate. There is a ring monomorphism, introduced already by Burnside,

$$\chi : \mathcal{A}(G)_{(p)} \to \prod_{[H]\in[S(G)]} \mathbb{Z}_{(p)}, \qquad [K]\mapsto \sum_{[H]\in[S(G)]} \big|(G/K)^H\big|\cdot[H].$$

The inclusion $[S_p^{p\text{-cent}}(G)]\subseteq[S_p(G)]$ gives rise to a composite ring homomorphism

$$\tilde{\Psi}: \mathcal{A}(G;p)_{(p)} \xrightarrow{\text{incl}} \mathcal{A}(G)_{(p)} \xrightarrow{\chi} \prod_{[S_p(G)]} \mathbb{Z}_{(p)} \xrightarrow{\text{proj}} \prod_{[S_p^{p\text{-cent}}(G)]} \mathbb{Z}_{(p)}.$$

Since $S_p^{\neg p\text{-cent}}(G)$ is closed to formation of subgroups, if $K \in S_p^{\neg\text{cent}}(G)$ and $H \in S_p^{p\text{-cent}}(G)$ then $(G/K)^H$ is empty. Hence, $\mathcal{A}^{\neg p\text{-cent}}(G)_{(p)}$ is contained in the kernel of $\tilde{\Psi}$ and there results a ring homomorphism

$$\Psi: \mathcal{A}^{p\text{-cent}}(G)_{(p)} \to \prod_{[Q]\in[S_p^{p\text{-cent}}(G)]} \mathbb{Z}_{(p)}, \qquad [P] \mapsto \sum_{[Q]\in[S_p^{p\text{-cent}}(G)]} \left|G/P^Q\right| \cdot [Q]. \tag{1}$$

For subgroups $H, K \leqslant G$ consider now

$$N_G(H,K) = \left\{g \in G\colon\ gHg^{-1} \leqslant K\right\}.$$

Clearly K acts on $N_G(H,K)$ by left translations and $N_G(H)$ acts on $N_G(H,K)$ by right translations. Clearly, the action of H on $K\backslash N_G(H,K)$ is trivial.

By construction of $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)$, see Section 2, we see that $\mathcal{C}(P,Q) = Q\backslash N_G(P,Q)/C_G(P)$ for any $P, Q \leqslant S$ which are $\mathcal{F}$-centric. As usual we let $[\mathcal{C}]$ denote the set of the isomorphism classes of the objects of $\mathcal{C}$.

Claim 1. *If P, Q are $\mathcal{F}$-centric subgroups of S then $|G/Q^P| = |C'_G(P)| \cdot |\mathcal{C}(P,Q)|$.*

Proof. By inspection

$$G/Q^P = N_G(P,Q)^{-1}/Q \approx Q\backslash N_G(P,Q).$$

Note that P is p-centric in G because it is $\mathcal{F}$-centric. Thus, $C_G(P) = C'_G(P) \times Z(P)$ and since $Z(P)$ acts trivially on $Q\backslash N_G(P,Q)$ it follows that

$$\mathcal{C}(P,Q) = Q\backslash N_G(P,Q)/C'_G(P).$$

Furthermore, the action of $C'_G(P)$ on $Q\backslash N_G(P,Q)$ is free because for any $x \in C'_G(P)$ and any $Qg \in Q\backslash N_G(P,Q)$, if $Qgx = Qg$ then $gxg^{-1} \in Q$, which implies $x = 1$ because x has order prime to p. Therefore $|\mathcal{C}(P,Q)| = |Q\backslash N_G(P,Q)| \cdot |C'_G(P)|$ and the result follows. □

By construction $P, Q \leqslant S$ are $\mathcal{F}$-conjugate if and only if they are conjugate in G. Also, $P \leqslant S$ is $\mathcal{F}$-centric if and only if it is p-centric. It follows that there is a natural one-to-one correspondence between the sets $[S_p^{\text{cent}}(G)]$ and $[\mathcal{C}]$. There results an isomorphism of free $\mathbb{Z}_{(p)}$-modules

$$\lambda: \mathcal{A}(\mathcal{F})_{(p)} \to \mathcal{A}^{p\text{-cent}}(G)_{(p)}$$

which is the identity on basis elements under the identification $[\mathcal{C}] = [S_p^{\text{cent}}(G)]$. Clearly λ is an isomorphism of $\mathbb{Z}_{(p)}$-modules (but it is not a ring homomorphism).

Let η denote the following element in $\prod_{[Q]\in[S_p^{\mathrm{cent}}(G)]}\mathbb{Z}_{(p)}$

$$\eta = \sum_{[Q]\in[S_p^{p\text{-cent}}(G)]} \left|C'_G(Q)\right| \cdot [Q]. \tag{2}$$

It follows from the definition of Ψ in (1), from Claim 1 and from the definition of the ring homomorphism Φ in (3.1) that the following square of $\mathbb{Z}_{(p)}$-modules is commutative. (Note: this is not a commutative square of rings!)

$$\begin{tikzcd}
\mathcal{A}(\mathcal{F})_{(p)} \arrow[r, "\lambda", "\cong"'] \arrow[d, "\Phi_{(p)}"'] & \mathcal{A}^{p\text{-cent}}(G)_{(p)} \arrow[d, "\Psi"] \\
\prod\limits_{[Q]\in[S_p^{\mathrm{cent}}(G)]} \mathbb{Z}_{(p)} \arrow[r, "\cong", "\cdot\eta"'] & \prod\limits_{[Q]\in[S_p^{\mathrm{cent}}(G)]} \mathbb{Z}_{(p)}
\end{tikzcd} \tag{3}$$

The arrow at the bottom is induced by multiplication by η which is an isomorphism of $\mathbb{Z}_{(p)}$-modules because the $|C'_G(Q)|$'s are invertible in $\mathbb{Z}_{(p)}$.

Claim 2. $\eta \in \operatorname{Im}\Phi_{(p)}$.

Proof. We apply Theorem 3.4, that is, we show that for every $K \leqslant S$ which is $\mathcal{F}$-centric (equivalently, K is p-centric) and fully normalized in $\mathcal{F}$, the following congruence hold

$$\sum_{[H]\in[\mathcal{C}]} n(K,H)\cdot\left|C'_G(H)\right| = 0 \mod \left(\left|\mathrm{Out}_S(K)\right|\right).$$

In this sum only the terms where, up to $\mathcal{F}$-conjugacy, $K \leqslant H \leqslant N_S(K)$ and H/K is cyclic show up because the numbers $n(K,H)$ vanish for all other H's. Now fix some $s \in N_S(K)$. Clearly $N_S(K)$ normalizes $C_G(K)$ and it therefore normalizes the characteristic subgroup $C'_G(K)$. Thus, $N_S(K)/K$ acts via conjugation on $C'_G(K)$ and for any $s \in N_S(K)$ the group $\langle s, K\rangle$ is p-centric because it contains K. Moreover,

$$C'_G\big(\langle s,K\rangle\big) = C_G\big(\langle s,K\rangle\big) \cap C'_G(K) = C'_G(K)^s,$$

namely, these are the fixed points of s in its action on $C'_G(K)$. Using Frobenius's formula,

$$\begin{aligned}
&\sum_{H\in[\mathcal{C}]} n(K,H)\cdot\left|C'_G(H)\right| \\
&\quad= \sum_{H\in[\mathcal{C}]}\Bigg(\sum_{[s]\in N_S(K)/K,\,\langle s,K\rangle\simeq_{\mathcal{F}}H} \left|C'_G(H)\right|\Bigg) \\
&\quad= \sum_{H\in[\mathcal{C}]}\Bigg(\sum_{[s]\in N_S(K)/K,\,\langle s,K\rangle\simeq_{\mathcal{F}}H} \left|C'_G\big(\langle s,K\rangle\big)\right|\Bigg)
\end{aligned}$$

$$= \sum_{[s]\in N_S(K)/K} \left|C'_G(\langle s, K\rangle)\right| = \sum_{[s]\in N_S(K)/K} \left|C'_G(K)^{[s]}\right|$$

$$= \left|N_S(K)/K\right| \cdot \left|\left\{\text{orbits of } N_S(K)/K \text{ on } C'_G(K)\right\}\right| \equiv 0 \mod \left|\mathrm{Out}_S(K)\right|.$$

We conclude from Theorem 3.4 that $\eta \in \mathrm{Im}\,\phi$. $\square$

Let A denote the image of $\Phi_{(p)}$ and B denote the image of Ψ. Both are $\mathbb{Z}_{(p)}$-submodules of $M = \prod_{[S_p^{\mathrm{cent}}(G)]} \mathbb{Z}_{(p)}$. Diagram (3) shows that $B = \eta \cdot A$. Since A is a subring of M, Claim 2 implies that $\eta \cdot A \subseteq A$. Multiplication with η therefore gives rise to a morphism of short exact sequences of $\mathbb{Z}_{(p)}$-modules

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot\eta"'] & M \arrow[r] \arrow[d, "\cdot\eta"', "\cong"] & M/A \arrow[r] \arrow[d, "\cdot\eta"] & 0 \\ 0 \arrow[r] & A \arrow[r] & M \arrow[r] & M/A \arrow[r] & 0 \end{tikzcd}$$

It follows that $M/A \xrightarrow{\cdot\eta} M/A$ is an epimorphism. By Corollary 3.5 M/A is a finite p-group and $\Phi_{(p)}$ is a monomorphism. It follows that $M/A \xrightarrow{\cdot\eta} M/A$ is an isomorphism and that Ψ is a ring monomorphism. Application of the five lemma now shows that $A \xrightarrow{\cdot\eta} A$ is an isomorphism. In particular $\eta \cdot A = A$. Since $\Phi_{(p)}$ and Ψ are ring monomorphisms $\mathcal{A}(\mathcal{F})_{(p)} \cong A = \eta \cdot A = B \cong \mathcal{A}^{p\text{-cent}}(G)_{(p)}$. $\square$

4. Examples

The Burnside ring of a finite group is an algebraic invariant which does not characterize the isomorphism type of the group: Thévenaz constructed in [13] two non-isomorphic groups $G_1 \ncong G_2$ with isomorphic Burnside rings.

The situation for fusion systems is similar: consider the 2-group $S = (\mathbb{Z}_2)^9$ and its automorphism group $\mathrm{GL}_9(2)$. As the symmetric group Σ_9 acts faithfully on S there are subgroups $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ in $\mathrm{GL}_9(2)$. Now consider the fusion systems $\mathcal{F}_1 = \mathcal{F}_S(S \rtimes \mathbb{Z}_9)$ and $\mathcal{F}_2 = \mathcal{F}_S(S \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_3))$. These two saturated fusion systems are not isomorphic as $\mathrm{Out}_{\mathcal{F}_1}(S) = \mathbb{Z}_9 \ncong \mathbb{Z}_3 \times \mathbb{Z}_3 = \mathrm{Out}_{\mathcal{F}_2}(S)$. Moreover, since S is abelian, the only $\mathcal{F}_i$-centric group for $i = 1, 2$ is S itself. Thus the matrix defining the monomorphism (3.1) becomes the scalar $|\mathrm{Out}_{\mathcal{F}_i}(S)|$ for $i = 1, 2$. As $|\mathrm{Out}_{\mathcal{F}_1}(S)| = 9 = |\mathrm{Out}_{\mathcal{F}_2}(S)|$ we deduce that $\mathcal{A}(\mathcal{F}_1)_{(2)} \cong \mathcal{A}(\mathcal{F}_2)_{(2)}$.

Also notice that the fusion systems of the groups $G_1 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ and $G_2 = \mathbb{Z}_9 \rtimes \mathbb{Z}_2$ at the prime 3 show that we can have isomorphic Burnside rings even with different order Sylow subgroups because $\mathcal{A}(\mathcal{F}_{\mathbb{Z}/3}(G_1))_{(3)} \cong \mathbb{Z}_{(3)} \cong \mathcal{A}(\mathcal{F}_{\mathbb{Z}/9}(G_2))_{(3)}$.

Here are more examples.

4.1. Lemma. *Let $\mathcal{F}$ be a saturated fusion system over S. Assume that $\mathcal{F}$ has exactly $n \geqslant 1$ conjugacy classes of $\mathcal{F}$-centric subgroup $P \leqslant S$ all of which have index p or 1 in S. Then $\mathcal{A}(\mathcal{F})_{(p)}$ is isomorphic to the subring of $\mathbb{Z}_{(p)}^n$ whose $\mathbb{Z}_{(p)}$-basis is $p \cdot e_1, \dots, p \cdot e_{n-1}$ and $e_1 + \cdots + e_n$ where e_i are the standard basis vectors.*

Proof. Let $P_1, \dots, P_{n-1}$ be representatives for the $\mathcal{F}$-conjugacy classes of the $\mathcal{F}$-centric subgroups of S of index p. Let $\mathcal{C}$ denote the category $\mathcal{O}(\mathcal{F}^c)$ and let $\mathcal{C}(P_i, S)$ denote the set of morphisms $\mathrm{Hom}_{\mathcal{O}(F^c)}(P_i, S)$. The matrix representing $\Phi_{(p)} : \mathcal{A}_{(p)}(\mathcal{F}) \to \mathbb{Z}^n_{(p)}$ has the form

$$\begin{pmatrix} |\mathrm{Out}_{\mathcal{F}}(P_1)| & & 0 & |\mathcal{C}(P_1, S)| \\ & \ddots & & \vdots \\ & & |\mathrm{Out}_{\mathcal{F}}(P_{n-1})| & |\mathcal{C}(P_{n-1}, S)| \\ 0 & & & |\mathrm{Out}_{\mathcal{F}}(S)| \end{pmatrix}.$$

Thus $\mathcal{A}(\mathcal{F})_{(p)}$ is isomorphic to the image of this matrix, namely the submodule of $\mathbb{Z}^n_{(p)}$ generated by its columns. Since P_i is $\mathcal{F}$-centric and $|S : P_i| = p$, it follows that $|\mathrm{Out}_{\mathcal{F}}(P_i)| = p\zeta_i$ where ζ_i is a unit in $\mathbb{Z}_{(p)}$. In particular the submodule U of $\mathbb{Z}^n_{(p)}$ spanned by $\{p \cdot e_1, \dots, p \cdot e_{n-1}\}$ is contained in the image of $\Phi_{(p)}$. Proposition 2.8 and the fact that $|\mathrm{Out}_{\mathcal{F}}(S)|$ is a unit in $\mathbb{Z}_{(p)}$ now imply that the last column of the matrix above is equal modulus U to the column vector $e_1 + \dots + e_n$ and therefore the image of $\Phi_{(p)}$ is equal to the submodule generated by U and $e_1 + \dots + e_n$. □

4.2. Example. Lemma 4.1 implies that if $\mathcal{F}$ is a fusion system over S where $|S| = p^3$ then $\mathcal{A}(\mathcal{F})_{(p)}$ depends only on the number of conjugacy classes of the $\mathcal{F}$-centric subgroups because no subgroup of order p can be $\mathcal{F}$-centric.

This gives a hassle-free calculation of the rings $\mathcal{A}(\mathcal{F})_{(p)}$ of the fusion systems on the extraspecial group p^{1+2}_+ where p is odd—all of which were classified by Ruiz and Viruel in [11]. For example, the ring $\mathcal{A}(\mathcal{F})_{(7)}$ of the exotic examples at the prime 7 listed in rows 8 and 11 of Table 1.2 in [11] are isomorphic to the 7-local Burnside ring of the fusion system of Fi_{24} at the prime 7. The exotic example appearing in the 12th row of this table has a 7-local Burnside ring whose underlying $\mathbb{Z}_{(7)}$-module has rank 2, and no other fusion system on 7^{1+2}_+ has an isomorphic Burnside ring.

Recall that if $\mathcal{F}_1$ is a sub fusion system of $\mathcal{F}_2$ over the same S then any $\mathcal{F}_2$-centric subgroup is also $\mathcal{F}_1$-centric.

4.3. Proposition. *Let S be a finite p-group and let $\mathcal{F}_1 \subseteq \mathcal{F}_2$ be saturated fusion systems on S. Assume that the following hold for any $P \leqslant S$ which is $\mathcal{F}_2$-centric.*

(i) *The conjugacy class of P in $\mathcal{F}_2$ is equal to its conjugacy class in $\mathcal{F}_1$.*
(ii) $|\mathrm{Aut}_{\mathcal{F}_2}(P) : \mathrm{Aut}_{\mathcal{F}_1}(P)| = 1 \bmod |R|$ *where $R \leqslant \mathrm{Out}_{\mathcal{F}_1}(P)$ is a Sylow p-subgroup. Moreover, if $\mathrm{Aut}_{\mathcal{F}_1}(P) \neq \mathrm{Aut}_{\mathcal{F}_2}(P)$ then P is minimal with respect to the property that it is $\mathcal{F}_2$-centric.*

Then the $\mathbb{Z}_{(p)}$-submodule I of $\mathcal{A}(\mathcal{F}_1)_{(p)}$ generated by the elements $[P]$ such that $P \leqslant S$ is $\mathcal{F}_1$-centric but not $\mathcal{F}_2$-centric, is an ideal in $\mathcal{A}(\mathcal{F}_1)_{(p)}$ and moreover, $\mathcal{A}(\mathcal{F}_2)_{(p)} \cong \mathcal{A}(\mathcal{F}_1)_{(p)}/I$.

Proof. Let C_1 and C_2 denote the sets of the conjugacy classes of the $\mathcal{F}_1$-centric and $\mathcal{F}_2$-centric subgroups of S. Moreover, let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote the categories $\mathcal{O}(\mathcal{F}_1^c)$ and $\mathcal{O}(\mathcal{F}_2^c)$ respectively. By definition I is the $\mathbb{Z}_{(p)}$-submodule of $\mathcal{A}_{(p)}(\mathcal{F}_1)$ generated by the elements $[P] \in C_1 \backslash C_2$. Consider the ring monomorphisms (3.1)

$$\Phi_1 : \mathcal{A}_{(p)}(\mathcal{F}_1) \to \mathbb{Z}_{(p)}^{C_1}, \qquad \Phi_2 : \mathcal{A}_{(p)}(\mathcal{F}_2) \to \mathbb{Z}_{(p)}^{C_2}.$$

By hypothesis (i) there is a natural inclusion $C_2 \subseteq C_1$, whence a ring epimorphism

$$\pi : \mathbb{Z}_{(p)}^{C_1} \to \mathbb{Z}_{(p)}^{C_2}.$$

Claim. $I = \ker(\pi \circ \Phi_1)$.

Proof. Observe that if $[P] \in C_1 \setminus C_2$ then $\mathcal{C}_1(Q, P)$ is empty if $[Q] \in C_2$ (otherwise P must be $\mathcal{F}_2$-centric). It follows immediately that $I \subseteq \ker(\pi \circ \Phi_1)$.

Conversely consider an element $x = \sum_{[P] \in C_1} \alpha_P [P]$ in $\mathcal{A}(\mathcal{F}_1)$ and assume that it is not in I. Then $\alpha_Q \neq 0$ for some $[Q] \in C_2$ and we choose Q of maximal order with this property. Recall that $\Phi_1(x)$ is a function $C_1 \to \mathbb{Z}_{(p)}$ and the maximality of P implies that

$$\Phi_1(x)\big([Q]\big) = \sum_{[P]} \alpha_P \cdot \big|\mathcal{C}_1(Q, P)\big| = \alpha_Q \cdot \big|\mathrm{Out}_{\mathcal{F}_1}(Q)\big| \neq 0.$$

This shows that $x \notin \ker(\pi \circ \Phi_1)$. We deduce that $I = \ker(\pi \circ \Phi_1)$. $\square$

From the claim it follows that $I \lhd \mathcal{A}_{(p)}(\mathcal{F}_1)$ and that $\mathrm{Im}(\pi \circ \Phi_1) \cong \mathcal{A}_{(p)}(\mathcal{F}_1)/I$. Clearly $\mathrm{Im}(\pi \circ \Phi_1)$ is a subring of $\mathbb{Z}_{(p)}^{C_2}$ and it remains to prove that it is equal to the image of Φ_2 which is isomorphic to $\mathcal{A}_{(p)}(\mathcal{F}_2)$ by Corollary 3.5.

Let $P_1, \ldots, P_k$ be representatives for the $\mathcal{F}_2$-conjugacy classes of minimal $\mathcal{F}_2$-centric subgroups of S. By hypothesis (ii)

$$\big|\mathrm{Out}_{\mathcal{F}_2}(P_i)\big| = \zeta_i \cdot \big|\mathrm{Out}_{\mathcal{F}_1}(P_i)\big|$$

where $\zeta_i = 1 \bmod |R_i|$ where R_i is a Sylow p-subgroup of $\mathrm{Out}_{\mathcal{F}_1}(P_i)$ and also of $\mathrm{Out}_{\mathcal{F}_2}(P_i)$. The equality of the $\mathcal{F}_1$- and $\mathcal{F}_2$-conjugacy classes of P_i together with Proposition 3.2 also implies that for any $\mathcal{F}_2$-centric $Q \leqslant S$

$$\big|\mathcal{C}_2(P_i, Q)\big| = \zeta_i \cdot \big|\mathcal{C}_1(P_i, Q)\big|. \tag{1}$$

For every P_i we consider $f_i = \pi \circ \Phi_1([P_i])$ and $g_i = \Phi_2([P_i])$. The minimality of P_i implies that $f_i([Q]) = |\mathrm{Out}_{\mathcal{F}_1}(P_i)|$ if $[Q] = [P_i]$ and it is zero otherwise. Similarly $g_i([Q]) = |\mathrm{Out}_{\mathcal{F}_2}(P_i)|$ if $[Q] = [P_i]$ and it is zero otherwise. Since R_i is a Sylow p-subgroup in both $\mathrm{Out}_{\mathcal{F}_1}(P_i)$ and $\mathrm{Out}_{\mathcal{F}_2}(P_i)$ we see that the $\mathbb{Z}_{(p)}$-submodules U of $\mathbb{Z}_{(p)}^{C_2}$ generated by $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ is equal to the submodule generated by $|R_1| \cdot e_{[P_1]}, \ldots, |R_k| \cdot e_{[P_k]}$ where $e_{[P_i]}$ are the obvious standard-basis elements in $\mathbb{Z}_{(p)}^{C_2}$.

Now consider any $[P] \in C_2$ which is not minimal. We now show that $f := \pi \circ \Phi_1([P])$ and $g := \Phi_2([P])$ are equal modulus U. Set $f = \pi \circ \Phi_1([P])$ and $g = \Phi_2([P])$. Given any $[Q] \in C_2$ observe that if Q is not minimal $\mathcal{F}_2$-centric then $\mathcal{C}_1(Q, P) = \mathcal{C}_2(Q, P)$ by hypothesis (ii) and Alperin's fusion theorem. Thus, by definition $f([Q]) - g([Q]) = 0$. We deduce that the support of $f - g$ is contained in $\{[P_1], \ldots, [P_k]\}$.

Now fix some $[P_i]$. Then from (1) we deduce that

$$f([P_i]) - g([P_i]) = |\mathcal{C}_1(P_i, P)| - |\mathcal{C}_2(P_i, P)|$$
$$= (1 - \zeta_i) \cdot |\mathcal{C}_1(P_i, P)| = 0 \mod |R_i|.$$

It follows immediately that $f - g \in U$. This completes the proof that $\mathrm{Im}(\Phi_2) = \mathrm{Im}(\pi \circ \Phi_1)$. $\square$

4.4. Example. We recall from [3, Proposition 5.3] a useful method to construct saturated fusion systems. We start with any fusion system $\mathcal{F}_S(G)$ of a finite group and fix subgroup $Q_1, \dots, Q_m$ of S such that Q_i is not subconjugate to Q_j if $i \neq j$. To avoid triviality we assume that $Q_i \neq S$. We set $K_i \leqslant \mathrm{Out}_G(Q_i)$ and fix $\Delta_i \leqslant \mathrm{Out}(Q_i)$ which contain K_i. We assume that $p \nmid |\Delta_i : K_i|$. We also assume that Q_i is p-centric in G but for any $P \lneq Q_i$ there exists some $\alpha \in \Delta_i$ such that $\alpha(P)$ is not p-centric in G. Furthermore, we assume that for any $\alpha \in \Delta_i \backslash K_i$ the order of $K_i \cap K_i^\alpha \leqslant \Delta_i$ is prime to p. Then the fusion system $\mathcal{F}$ generated by $\mathcal{F}_S(G)$ and $\mathcal{F}_{Q_i}(\Delta_i)$ is saturated.

This method of construction was introduced in [4, Proposition 5.1]. It yields many exotic examples, e.g. [2, Example 9.3], [4], [11, Table 1.2] and [5]. We claim that in all these cases Proposition 4.3 applies to the inclusion $\mathcal{F}_S(G) \leqslant \mathcal{F}$.

To see this observe that the conditions under which the construction of $\mathcal{F}$ is carried out guarantee that the Q_i's are minimal $\mathcal{F}$-centric subgroups. Thus, the $\mathcal{F}$-centric subgroups are the $\mathcal{F}_S(G)$-centric subgroups which are not subconjugate to one of the Q_i's. From the construction it is clear that the $\mathcal{F}$-conjugacy class of any $\mathcal{F}$-centric $P \leqslant S$ is equal to its conjugacy class in $\mathcal{F}_S(G)$. It is also clear that only the automorphism groups of the Q_i's are altered in the passage from $\mathcal{F}_S(G)$ to $\mathcal{F}$ and they are minimal $\mathcal{F}$-centric by construction. Thus it only remains to prove that $|\Delta_i : K_i| = 1 \mod |R_i|$ where R_i is a Sylow p-subgroup of K_i.

Let R_i act on the set of cosets Δ_i / K_i via left translation. If $\alpha \in \Delta_i \setminus K_i$ then $R_i \cap \alpha K_i \alpha^{-1}$ is trivial because by hypothesis the order of $K_i \cap \alpha K_i \alpha^{-1}$ is prime to p. It follows that R_i fixes the coset eK_i and acts freely on the remaining cosets. Therefore $|\Delta_i : K_i| = 1 \mod |R_i|$.

This shows that all the conditions of Proposition 4.3 are fulfilled for the inclusion $\mathcal{F}_S(G) \leqslant \mathcal{F}$ and therefore $\mathcal{A}_{(p)}(\mathcal{F})$ is a quotient ring of $\mathcal{A}_{(p)}(\mathcal{F}_S(G))$.

For example, all the fusion systems listed in the [2, Example 9.3] have the same p-local Burnside ring as the fusion system of the groups $\Gamma \rtimes A$ appearing in their construction.

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